A General Theorem on an Optimum Stepped Impedance Transformer*

HENRY J. RIBLET†

Summary—With the assistance of a mathematical theorem demonstrated by Eaton in a companion paper, it is shown rigorously, in the limit of small impedance transformation, that the familiar binomial impedance transformer, consisting of equal quarter-wave steps, is the shortest, monotonic, maximally-flat, stepped, transmission-line transformer having steps commensurate in length with the midband guide-wavelength, and coincident zeros at the midband frequency.

It is shown how this theorem places very severe limitations on any effort to improve on the performance of a quarter-wave transformer by increasing the number of its impedance steps without a corresponding increase in its length.

Introduction

L. SOLYMAR[1] has, in a recent paper, considered the problem of the optimum design of monotonic, stepped, transmission-line transformers. He has made the simplifying assumption that multiple reflections from the impedance discontinuities can be neglected, and has introduced the requirement of monotonicity to avoid the problem of "supermatch" which can otherwise appear, even when multiple reflections are considered. He employs the even polynomials proposed by Riblet,[2] to construct examples which show the interesting fact that, for given relative bandwidth, the quarter-wave transformer does not give the smallest pass-band reflection coefficient if additional length is available. He observes that, with transformers less than one-eighth wavelength long, this procedure results in nonmonotonic solutions.

Solymar's problem involves the length of the transformer, its bandwidth, and the ratio of tolerable reflection coefficient in the pass band to the reflection coefficient to be transformed. Any general discussion of the optimum design is exceedingly involved, and the results will certainly depend on the constraints placed on these three variables.

Several years ago, the writer considered the same general problem and was forced by its analytical difficulty to limit his investigation to the maximally-flat transformer. This restriction, however, permits a major simplification in the statement of the problem, since the variables of bandwidth and tolerable reflection coefficient may then be simultaneously specified, while the number of coincident zeros at the "resonant" frequency precisely determines the length of the transformer.

Even for an impedance transformer of finite bandwidth, of course, its tolerable VSWR is closely related to the number of zeros occurring in the operating band. Consideration of Solymar's examples will show that the number of zeros in the band of his "optimum" transformers does not exceed the number available from the longest quarter-wave transformer which can be fitted into the available length. When he improves on the performance that a quarter-wave transformer will yield, it is the spacing of the zeros which is adjusted. His efforts to introduce additional zeros leads to nonmonotonic solutions. In fact, it may be conjectured, in general, that additional zeros in the operating band of the impedance transformer will result in a nonmonotonic design.

The Problem

Consider the stepped, transmission-line transformer shown schematically in Fig. 1, operating between input

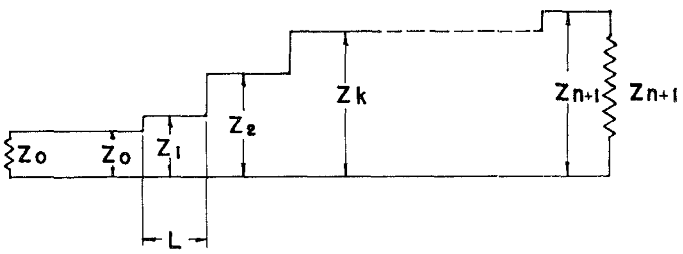

Fig. 1—Schematic of n-section transformer.

and output impedances Z_0 and Z_{n+1}, respectively. Γ_i is the reflection coefficient at the ith step. For a monotonic transformer all of the Γ_i have the same sign. The input reflection coefficient ρ neglecting multiple reflections, is known to be

$$\rho = \Gamma_1 + \Gamma_2 x + \Gamma_3 x^2 + \cdots + \Gamma_{n+1} x^n,$$

where $x = \exp\{i4\pi l/\lambda_g\}$.

If we require that ρ have n zeros at the particular frequency, where $\lambda_g = \bar{\lambda}_g$, we immediately find a solution of the problem in the form

$$\rho = C(x+1)^n, \tag{1}$$

if we select $l = \bar{\lambda}_g/4$, since then $x = \exp\{i\pi\bar{\lambda}_g/\lambda_g\}$ which equals -1 at the particular frequency. This is, of course, the familiar binomial, maximally-flat, transformer consisting of n, equal-length quarter-wave sections in which

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† Microwave Development Labs., Inc., Wellesley, Mass.

[1] L. Solymar, "Some notes on the optimum design of stepped transmission-line transformers," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-6, pp. 374–378; October, 1958.

[2] H. J. Riblet, "Discussion on 'A current distribution for broadside arrays which optimizes the relationships between beamwidth and side-lobe level'," PROC. IRE, vol. 35, pp. 489–492; May, 1947.

the Γ_i's have the ratios of the binomial coefficients. For engineering purposes, (1) may be rewritten as

$$|\rho| = \Gamma\{\cos(\pi\bar{\lambda}_g/2\lambda_g)\}^n, \quad (2)$$

where Γ is the reflection coefficient to be matched by the transformer. Eq. (2) defines, in terms of n, the bandwidth over which a given $|\rho|/\Gamma$ will not be exceeded.

It is the object of this paper to indicate how the binomial, quarter-wave transformer is optimum in the sense that no shorter stepped, monotonic transformer can have additional zeros at the chosen frequency. For this purpose, consider a transformer consisting of sections each $\bar{\lambda}_g/4r$ in length, where r is an integer. If a shorter, monotonic transformer could be designed using sections of this length, still having n coincident zeros, it would mean that a polynomial of degree less than nr in $x=\exp\{i\pi\bar{\lambda}_g/r\lambda_g\}$ could be constructed having positive real coefficients, with n coincident roots for $\lambda_g=\bar{\lambda}_g$. The condition on the degree follows from the fact that the over-all length of an impedance transformer is equal to the degree of the polynomial representing the input reflection coefficient multiplied by the length of the individual transformer sections. That this is impossible follows immediately from the purely mathematical theorem.[3]

Theorem: The real polynomial of minimum degree with positive coefficients having n roots at $e^{i\pi/r}$ is $(x^r+1)^n$.

Moreover since the solution $\rho=C(x^r+1)^n$ for the shorter steps is identical to (1) for the quarter-wave steps, we see that our efforts to improve on the quarter-wave transformer have brought us back to the starting point. We can thus prove the following theorem on an optimum impedance transformer.

Theorem A: The shortest, monotonic, stepped impedance transformer, all of whose steps are commensurate in length with $\bar{\lambda}$ having n coincident zeros for $\lambda_g=\bar{\lambda}_g$, is the binomial, quarter-wave transformer having n steps.

Proof: The requirement that the steps be commensurate with $\bar{\lambda}_g$ permits the selection of an r sufficiently large so that any shorter transformer meeting the conditions of the theorem can be thought to consist of steps each $\bar{\lambda}_g/4r$ in length. The theorem proved by Eaton is then applicable and a contradiction results.

[3] This theorem was conjectured by the writer, but its truth was in doubt for over a year before the ingenious proof given in the companion paper was found by J. E. Eaton. During that year, a careful search of the literature and inquiry of experts in the related branch of analysis failed to reveal any previous interest in this type of problem.

Engineering Applications

This theorem cannot restrict the performance of practical transformers without additional arguments involving "continuity" and "limits" since there is no way of determining when the conditions requiring commensurate lengths and coincident zeros have been met.[4] Accordingly its rigorous application is limited to purely theoretical design procedures. For example, it may be applied to the problem of Solymar, as follows.

Theorem B: No design procedure, which is independent of bandwidth, can yield a monotonic, stepped, impedance transformer having n zeros in its pass band, consisting of a fixed number of sections of fixed length, each commensurate with $\bar{\lambda}_g$, which is shorter in over-all length than $n\bar{\lambda}_g/4$.

Proof: The existence of such a design procedure would imply the existence of an infinite sequence of polynomials of fixed degree having the property that n of their roots approach some limiting value $e^{i\pi/r}$. The coefficients of these polynomials each then constitute a finite number of infinite sequences of bounded, positive real numbers. By a fundamental theorem, these sequences have limit points which are non-negative and bounded, and, thus, define a positive, real polynomial with n coincident zeros. This polynomial must at least be of degree nr, and so the design procedure cannot yield transformers shorter than $n\bar{\lambda}_g/4$.

Conclusion

Two theorems are demonstrated showing that the quarter-wave transformer is optimum under certain idealized conditions. Although these conditions are implicit in present design procedures, a deficiency exists in the theory which has the result that it is not applicable to an actual transformer. This limitation is not essential to the theory, however, and one may hope that it will be removed ultimately.

In the meantime the theorems give mathematical reality to the demarcation between monotonic design and "supermatch" and will serve as beacons pointing to certain obstacles which will have to be faced in any effort to put more zeros in the pass band of a monotonic transformer than are contained in the quarter-wave transformer of optimum design.

[4] Although some progress has been made in this direction, the problem is complicated by its delicate algebraic nature. For example, a quarter-wave transformer when imagined to consist of shorter length steps is certainly on the verge of nonmonotonicity.